An Introduction to Bearing Rigidity-based Formation Control

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ECC 2022 WS – Rigidity Theory applied to Dynamic Systems: from Parallel Robots to Multi-Agent Formations

state-of-the-art

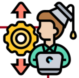
- S. Zhao, D. Zelazo. *Bearing Rigidity and Almost Global Bearing-only Formation Stabilization*, TAC2016
  ▷ control of formations acting in $\mathbb{R}^d$, $d \in \{2, 3\}$ (particle point agent model)
- D. Zelazo, P. R. Giordano, A. Franchi. *Bearing-Only Formation Control Using an SE(2) Rigidity Theory*. CDC2015
  ▷ control of formations acting in $\mathbb{R}^2 \times \mathbb{S}^1 = SE(2)$ (rigid body agent model)
- F. Schiano, A. Franchi, D. Zelazo, P. R. Giordano. *A Rigidity-Based Decentralized Bearing Formation Controller for Groups of Quadrotor UAVs*. IROS2016
  ▷ control of formations acting in $\mathbb{R}^3 \times \mathbb{S}^1$ (rigid body agent model)
- G. Michieletto, A.Cenedese. *Formation Control for Fully Actuated Systems: a Quaternion-based Bearing Rigidity Approach*. ECC2019
  ▷ control of formations acting in $SE(3)$ (rigid body agent model)

is there the possibility to treat the formation control problem in any domain?

PART I

General Framework and Unified View of Bearing Rigidity

- framework model for generic multi-agent formation
- rigidity properties of static and dynamic frameworks
- extended bearing rigidity matrix



PART II

Application to Multi-Agent Formations

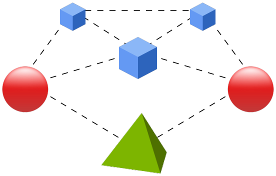
- homogeneous formations in $\mathbb{R}^d$, $\mathbb{R}^d \times \mathbb{S}^1$, $SE(3)$
- general properties for homogeneous formations
- other homogeneous and heterogeneous formations

PART I
General Framework and Unified View of Bearing Rigidity

Framework Formation Model

any i-th agent of the formation $i \in \{1 \dots n \geq 3\}$

- $\chi_i \in \mathcal{D}_i \subseteq SE(3)$ configuration (state + motion constraints)
- $\{\mathbf{b}_k = \mathbf{b}_{ij} \in \mathcal{M}_k \subseteq \mathbb{S}^2\}$ bearing measurements



Framework in $\bar{\mathcal{D}}$

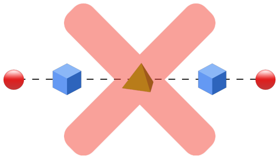
ordered pair $(\mathcal{G}, \chi)$ consisting of

- a (directed or undirected) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = n$
- a map $\chi : \mathcal{V} \to \bar{\mathcal{D}} = \prod_{i=1}^{n} \mathcal{D}_i \subseteq SE(3)^n$ s.t. $v_i \mapsto \chi(v_i) := \chi_i \in \mathcal{D}_i$

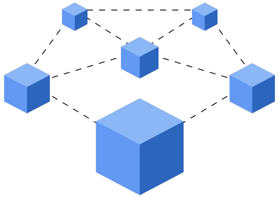
  $\triangleright$ $\chi = (\chi_1 \dots \chi_n) \in \bar{\mathcal{D}}$ *formation configuration*

Bearing Function

$$\mathbf{b}_{\mathcal{G}} : \bar{\mathcal{D}} \to \bar{\mathcal{M}} = \prod_{k=1}^{m} \mathcal{M}_k \subseteq \mathbb{S}^{2m}, \quad \chi \in \bar{\mathcal{D}} \to \mathbf{b}_{\mathcal{G}}(\chi) = \left[\mathbf{b}_1^\top \dots \mathbf{b}_m^\top\right]^\top \in \bar{\mathcal{M}} \quad \textit{meas vector}$$



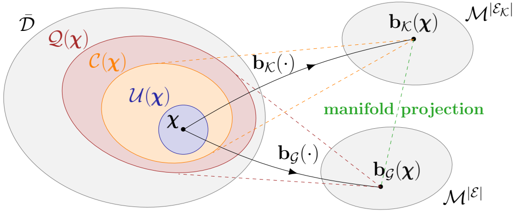
Noncolinear Formation
all agents are in distinct positions and do not lie along the same line in global frame



Homogeneous Formation
if $\mathcal{D}_i = \mathcal{D}\ \forall i \in \{1 \dots n\}$, hence $\bar{\mathcal{D}} = \mathcal{D}^n$
(and $\bar{\mathcal{M}} = \mathcal{M}^m$ with $\mathcal{M}_k = \mathcal{M}, \forall k \in \{1 \dots m\}$)

Static Frameworks: $(\mathcal{G}, \chi)$ with both $\mathcal{G}$ and χ time-invariant

Bearing Equivalence	**Bearing Congruence**
$(\mathcal{G}, \chi)$, $(\mathcal{G}, \chi')$ BE if $\mathbf{b}_{\mathcal{G}}(\chi) = \mathbf{b}_{\mathcal{G}}(\chi')$	$(\mathcal{G}, \chi)$, $(\mathcal{G}, \chi')$ BC if $\mathbf{b}_{\mathcal{K}}(\chi) = \mathbf{b}_{\mathcal{K}}(\chi')$
$\Downarrow$	$\Downarrow$
$\mathcal{Q}(\chi) = \mathbf{b}_{\mathcal{G}}^{-1}(\mathbf{b}_{\mathcal{G}}(\chi)) \subseteq \bar{\mathcal{D}}$	$\mathcal{C}(\chi) = \mathbf{b}_{\mathcal{K}}^{-1}(\mathbf{b}_{\mathcal{K}}(\chi)) \subseteq \mathcal{Q}(\chi)$
BE frameworks set	BC frameworks set



Bearing Rigidity in $\bar{\mathcal{D}}$

$(\mathcal{G}, \chi)$ BR if $\exists\, \mathcal{U}(\chi) \subseteq \bar{\mathcal{D}}$ s.t. $\mathcal{Q}(\chi) \cap \mathcal{U}(\chi) = \mathcal{C}(\chi) \cap \mathcal{U}(\chi)$

Global Bearing Rigidity in $\bar{\mathcal{D}}$

$(\mathcal{G}, \chi)$ GBR if $\mathcal{Q}(\chi) = \mathcal{C}(\chi)$

any GBR framework $(\mathcal{G}, \chi)$ is also BR

Dynamic Frameworks: $(\mathcal{G}, \chi)$ with $\mathcal{G}$ time-invariant and χ evolving in $\bar{\mathcal{D}}$

▷ $\chi = \chi(t) = (\chi_1(t) \dots \chi_n(t))$ s.t. $\frac{d\chi_i(t)}{dt} = f_i(\chi_i(t), \mathbf{u}_i)$

$\mathbf{u}_i \in \mathcal{I}_i$ variation (motion constraints)

$\mathbf{u} = \left[\mathbf{u}_1^\top \dots \mathbf{u}_n^\top\right]^\top \in \bar{\mathcal{I}} = \prod_{i=1}^n \mathcal{I}_i$ *variation vector*

Bearing Rigidity Matrix

$$\mathbf{B}_\mathcal{G}(\chi(t)) \quad \text{s.t.} \quad \dot{\mathbf{b}}_\mathcal{G}(\chi(t)) = \frac{d}{dt}\mathbf{b}_\mathcal{G}(\chi(t)) = \mathbf{B}_\mathcal{G}(\chi(t))\mathbf{u}$$

Infinitesimal Variation

$$\mathbf{u} \in \ker(\mathbf{B}_\mathcal{G}(\chi(t))) \subseteq \ker(\mathbf{B}_\mathcal{K}(\chi(t)))$$

Trivial Variation

$$\mathbf{u} \in \ker(\mathbf{B}_\mathcal{K}(\chi(t)))$$

Infinitesimal Bearing Rigidity in $\bar{\mathcal{D}}$

$$(\mathcal{G}, \chi) \text{ IBR if } \ker(\mathbf{B}_\mathcal{G}(\chi(t))) = \ker(\mathbf{B}_\mathcal{K}(\chi(t)))$$

$\bar{\mathcal{D}} \subseteq SE(3)^n$: any agent can be modeled as a **rigid body** with

- $c_i = c_i^t + c_i^r \in [0, 6]$ dofs ($c_i^t, c_i^r \in [0, 3]$ translational/rotational dofs)
- $\mathbf{p}_i(t) = \left[p_i^x(t)\ p_i^y(t)\ p_i^z(t)\right]^\top \in \mathbb{R}^3$ position in global frame
- $\mathbf{R}_i(t) = \mathbf{R}\left(\{\theta_{i,h}(t), \mathbf{e}_h\}_{h=1}^3\right) \in SO(3)$ attitude in global frame
- *local* bearing measurements $\Rightarrow$ $\mathcal{G}$ directed

$\mathbf{b}_{\mathcal{G}}^+(\chi(t)) = \mathrm{diag}(s_{ij}(t)\mathbf{R}_i^\top(t))\bar{\mathbf{E}}^\top \mathbf{p}(t) \in \mathbb{S}^{2m}$

$s_{ij} = \|\mathbf{p}_{ij}(t)\|^{-1} = \|\mathbf{p}_j(t) - \mathbf{p}_i(t)\|^{-1} \in \mathbb{R}$
$\bar{\mathbf{E}} = \mathbf{E} \otimes \mathbf{I}_d \in \mathbb{R}^{dn \times dm}$, $\mathbf{p}(t) \in \mathbb{R}^{3n}$

$\mathbf{u}^+ = \left[\mathbf{u}_p^\top \ \ \mathbf{u}_a^\top\right]^\top \in \mathbb{R}^{6n}$

$\mathbf{u}_p \in \mathbb{R}^{3n}$ agents position variation vector
$\mathbf{u}_a \in \mathbb{R}^{3n}$ agents attitude variation vector

Extended Bearing Rigidity Matrix

$$\mathbf{B}_{\mathcal{G}}^+(\chi(t)) \in \mathbb{R}^{3m \times 6n} \quad \text{s.t.} \quad \dot{\mathbf{b}}_{\mathcal{G}}^+(\chi(t)) = \frac{d}{dt}\mathbf{b}_{\mathcal{G}}^+(\chi(t)) = \mathbf{B}_{\mathcal{G}}^+(\chi(t))\mathbf{u}^+$$

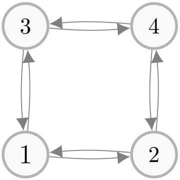
$$\mathbf{B}^{+}_{\mathcal{G}}(\chi(t)) = \begin{bmatrix} \mathbf{D}_p(t)\mathbf{U}\bar{\mathbf{E}}^\top & \mathbf{D}_a(t)\mathbf{V}\bar{\mathbf{E}}_o^\top \end{bmatrix} \in \mathbb{R}^{3m \times 6n}$$

where

- $\mathbf{D}_p(t), \mathbf{D}_a(t) \in \mathbb{R}^{3m \times 3m}$ are derived from the orthogonal projections of **relative position and attitude**
- $\mathbf{U} = \mathrm{diag}(\mathbf{U}_{ij}) \in \mathbb{R}^{3m \times 3m}$ and $\mathbf{V} = \mathrm{diag}(\mathbf{V}_{ij}) \in \mathbb{R}^{3m \times 3m}$ take into account the (time-invariant) matrices $\mathbf{U}_{ij}, \mathbf{V}_{ij} \in \mathbb{R}^{3 \times 3}$ defining, respectively, the **translational directions of the bearing measurement $\mathbf{b}_{ij}$** and i-th and j-th **agents rotation directions** in the 3D space wrt the i-th agent frame
- $\bar{\mathbf{E}} = \mathbf{E} \otimes \mathbf{I}_3, \bar{\mathbf{E}}_o = \mathbf{E}_o \otimes \mathbf{I}_3 \in \mathbb{R}^{3n \times 3m}$ are derived from the (time-invariant) **incidence matrix** of the graph $\mathcal{G}$.

$(\mathcal{G}, \chi)$ IBR **iff** $\mathrm{rk}(\mathbf{B}^{+}_{\mathcal{G}}(\chi(t))) = \mathrm{rk}(\mathbf{B}^{+}_{\mathcal{K}}(\chi(t)))$

example: $n = 4$ planar particle points
on the corner of a square with side 1m
$(c = c_i = c_i^t + c_i^r = 2 + 0 = 2)$



$$\mathbf{B}_{\mathcal{G}}^{+}(\chi(t)) = \begin{bmatrix} \mathbf{D}_p(t)\mathbf{U}\bar{\mathbf{E}}^\top & \mathbf{D}_a(t)\mathbf{V}\bar{\mathbf{E}}_o^\top \end{bmatrix} \in \mathbb{R}^{3m \times 6n}$$

$$\text{s.t.} \quad \dot{\mathbf{b}}_{\mathcal{G}}^{+}(\chi(t)) = \mathbf{B}_{\mathcal{G}}^{+}(\chi(t))\mathbf{u}^{+}$$

- $\mathbf{D}_p(t) = \mathrm{diag}\left(s_{ij}(t)\mathbf{R}_i^\top(t)\mathbf{P}\left(\bar{\mathbf{p}}_{ij} = \frac{\mathbf{p}_{ij}}{s_{ij}}\right)\right) \in \mathbb{R}^{3m \times 3m}$ $\quad \Rightarrow \; s_{ij}(t) = 1, \mathbf{R}_i(t) = \mathbf{I}_3$

  $= \mathrm{diag}\left(\mathbf{P}\left(\bar{\mathbf{p}}_{ij}\right)\right) = \mathrm{diag}\left(\mathbf{I}_3 - \begin{bmatrix} p_{ij}^x & p_{ij}^y & 0 \end{bmatrix}^\top \begin{bmatrix} p_{ij}^x & p_{ij}^y & 0 \end{bmatrix}\right)$

- $\mathbf{U} = \mathrm{diag}(\mathbf{U}_{ij}) \in \mathbb{R}^{3m \times 3m}$: translational directions of measurement $\mathbf{b}_{ij}$ $\quad \Rightarrow \; \mathbf{U}_{ij} = \begin{bmatrix} \mathbf{e}_1 \; \mathbf{e}_2 \; \mathbf{0}_3 \end{bmatrix}$

- $\mathbf{V} = \mathrm{diag}(\mathbf{V}_{ij}) \in \mathbb{R}^{3m \times 3m}$: i-th and j-th agents rotation directions $\quad \Rightarrow \; \mathbf{V}_{ij} = \mathbf{0}_{3 \times 3}$

PART II
Application to Multi-Agent Formations

Homogeneous Formations in $\mathbb{R}^d$, $\mathbb{R}^d \times \mathbb{S}^1$, $SE(3)$

all the studied homogeneous formations can be recast
in the unified bearing rigid framework

$\mathcal{D}$	χ_i		$\mathbf{b}_{ij}$	$\mathbf{U}_{ij}$	$\mathbf{V}_{ij}$
	$\mathbf{p}_i$	$\mathbf{R}_i$			
$SE(3)$	$\begin{bmatrix} p_i^x & p_i^y & p_i^z \end{bmatrix}^\top$	$\mathbf{R}\left(\{\theta_{i,h}(t), \mathbf{e}_h\}_{h=1}^3\right)$	$\mathbf{R}_i^\top \bar{\mathbf{p}}_{ij} \in \mathbb{S}^2$	$\mathbf{I}_3$	$\mathbf{I}_3$
$\mathbb{R}^3 \times \mathbb{S}^1$	$\begin{bmatrix} p_i^x & p_i^y & p_i^z \end{bmatrix}^\top$	$\mathbf{R}\left(\theta_i(t), \mathbf{v}\right), \mathbf{v} = \sum_{h=1}^{3} v_h \mathbf{e}_h$	$\mathbf{R}_i^\top \bar{\mathbf{p}}_{ij} \in \mathbb{S}^2$	$\mathbf{I}_3$	$[\mathbf{0}_{3\times 2} \; \mathbf{v}]$
$\mathbb{R}^2 \times \mathbb{S}^1$	$\begin{bmatrix} p_i^x & p_i^y & 0 \end{bmatrix}^\top$	$\mathbf{R}\left(\theta_i(t), \mathbf{e}_3\right)$	$\mathbf{R}_i^\top \bar{\mathbf{p}}_{ij} \in \mathbb{S}^1$	$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{0}_3 \end{bmatrix}$	$[\mathbf{0}_{3\times 2} \; \mathbf{e}_3]$
$\mathbb{R}^3$	$\begin{bmatrix} p_i^x & p_i^y & p_i^z \end{bmatrix}^\top$	$\mathbf{I}_3$	$\bar{\mathbf{p}}_{ij} \in \mathbb{S}^1$	$\mathbf{I}_3$	$\mathbf{0}_{3\times 3}$
$\mathbb{R}^2$	$\begin{bmatrix} p_i^x & p_i^y & 0 \end{bmatrix}^\top$	$\mathbf{I}_3$	$\bar{\mathbf{p}}_{ij} = \frac{\mathbf{p}_{ij}}{s_{ij}} \in \mathbb{S}^0$	$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{0}_3 \end{bmatrix}$	$\mathbf{0}_{3\times 3}$

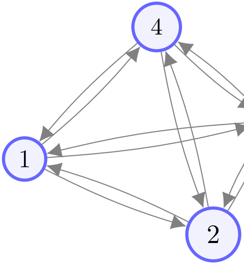
$$(\mathcal{G}, \chi(t)) \text{ with } c_i = c \leq 6 \text{ IBR} \quad \textbf{iif} \quad \text{rk}\left(\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))\right) = cn - c - 1$$

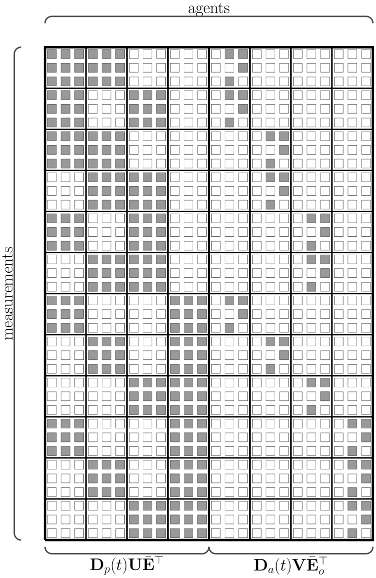
- for $(\mathcal{G}, \chi(t))$ in any $\bar{\mathcal{D}}$

$$\begin{array}{lcl} \text{GBR} & \Longrightarrow & \text{BR} \\ \text{BR} & \Longleftrightarrow & \text{IBR} \\ \text{GBR} & \Longrightarrow & \text{IBR} \end{array}$$

- for $(\mathcal{G}, \chi(t))$ in $\bar{\mathcal{D}} = \mathbb{R}^{dn}, d \in \{2, 3\}$

$$\begin{array}{lcl} \text{GBR} & \Longleftrightarrow & \text{BR} \\ \text{BR} & \Longleftrightarrow & \text{IBR} \\ \text{GBR} & \Longleftrightarrow & \text{IBR} \end{array}$$



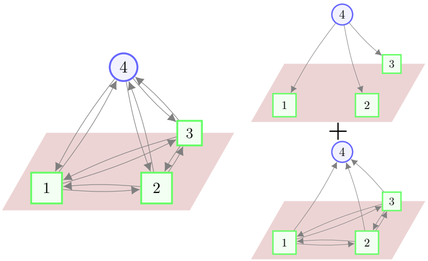


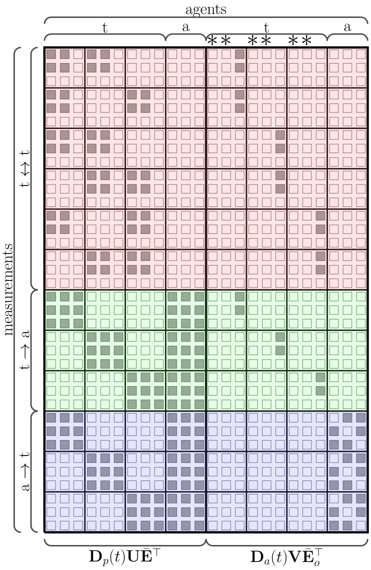
$n = 4$ aerial platforms with gimbal cameras (roll motion denied) $\rightarrow c_i = c = 5 = 3 + 2$

framework $(\mathcal{K}, \chi)$, $\chi_1 \dots \chi_4 \in \mathbb{R}^3 \times \mathbb{S}^2$

- ▷ $\mathbf{U}_{ij} = \mathbf{I}_3, \quad \mathbf{V}_{ij} = [\mathbf{0}_3 \; \mathbf{e}_2 \; \mathbf{e}_3]$
- ▷ $\text{rk}(\mathbf{B}_{\mathcal{K}}^+(\chi)) = cn - c - 1 = 14$

Heterogeneous Formations





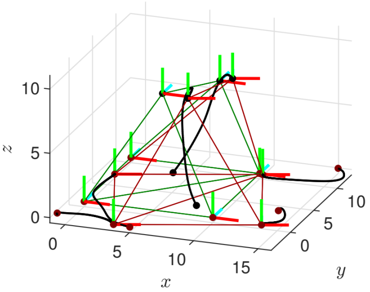
$n = 4 = 3$ unicycle-modeled terrestrial robots
$+1$ fully actuated aerial platform

fram. $(\mathcal{K}, \chi)$, $\chi_1 \dots \chi_3 \in \mathbb{R}^2 \times \mathbb{S}^1$, $\chi_4 \in SE(3)$

(i,j)	$\mathbf{U}_{ij}$	$\mathbf{V}_{ij}$
$(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)$	$[\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{0}_3]$	$[\mathbf{0}_{3\times 2}\ \mathbf{e}_3]$
$(1,4),(2,4),(3,4)$	$\mathbf{I}_3$	$[\mathbf{0}_{3\times 2}\ \mathbf{e}_3]$
$(4,1),(4,2),(4,3)$	$\mathbf{I}_3$	$\mathbf{I}_3$

Take-home message

FACT
bearing rigidity-based state-of-the-art formation controllers exploit rigidity matrix



stabilization control input

$$\mathbf{u} = k_c \mathbf{B}_{\mathcal{G}}(\chi)^\top \mathbf{b}_{\mathcal{G}}(\chi_d)$$

the proposed unified bearing rigidity framework allows to
model, analyze and CONTROL
both **homogenous and heterogenous** multi-agent formations
acting in **any domain**

Thank you for your time!
any questions?

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