Angle rigidity theory

convex polyhedra and robotic formation movement

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Joint work with Dr. Liangming Chen

Formation movement in nature

Fish schools [1]

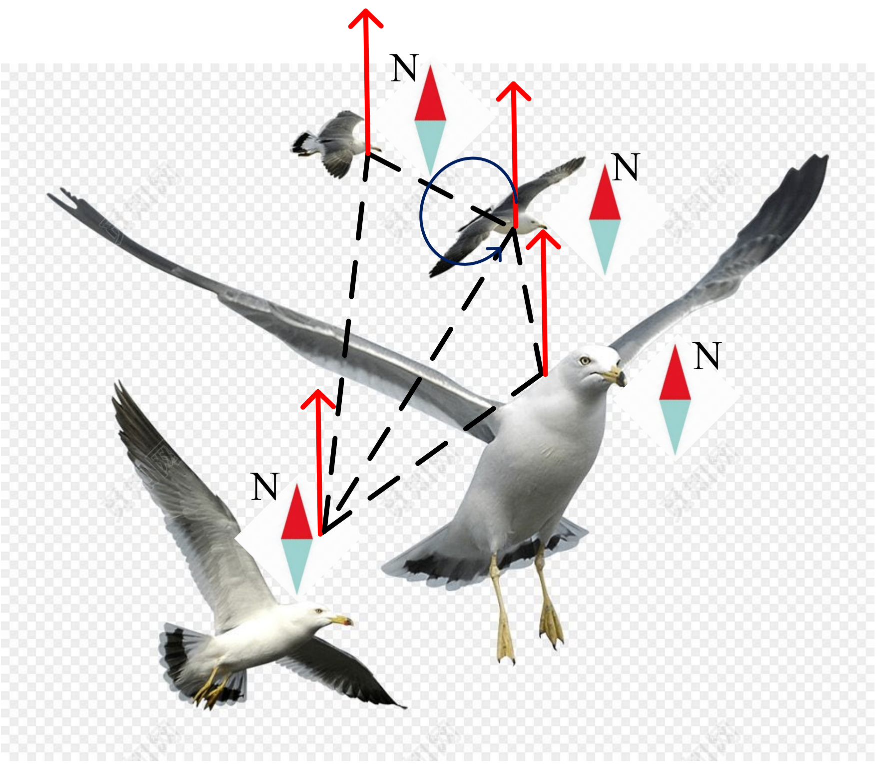
Bird flocks [2]

Sheep herds [3]

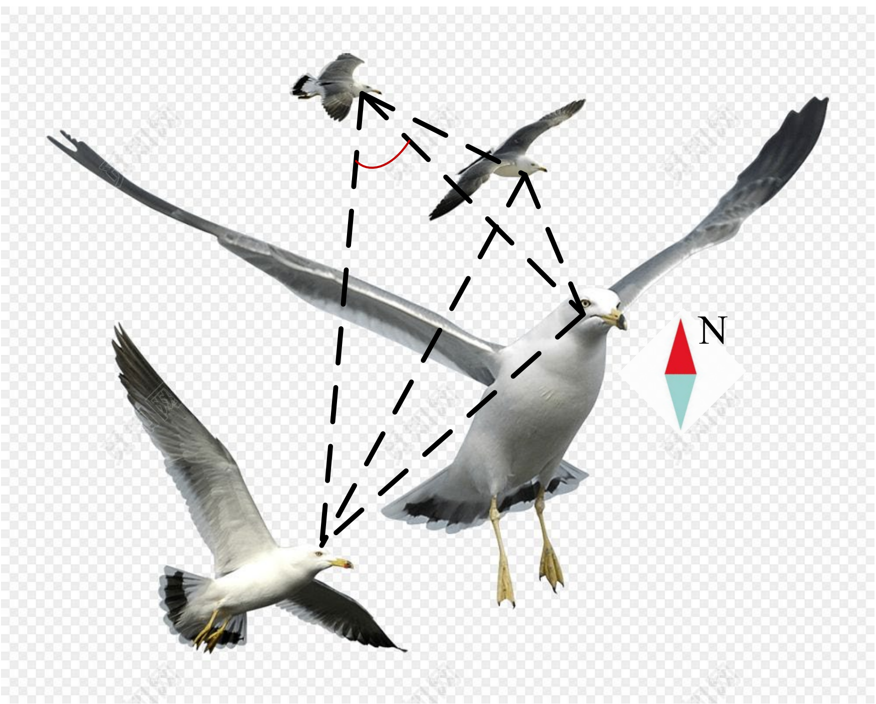
Locust swarms [4]

[1] https://www.pinterest.com/pin/514888169895507939/
[2] https://nickfrosst.github.io/flock_dynamics/
[3] https://new.qq.com/omn/20191009/20191009A0K7UZ00.html
[4] https://www.youtube.com/watch?v=6bx5JUGVahk

Formation maintenance under bearing/angle constraints



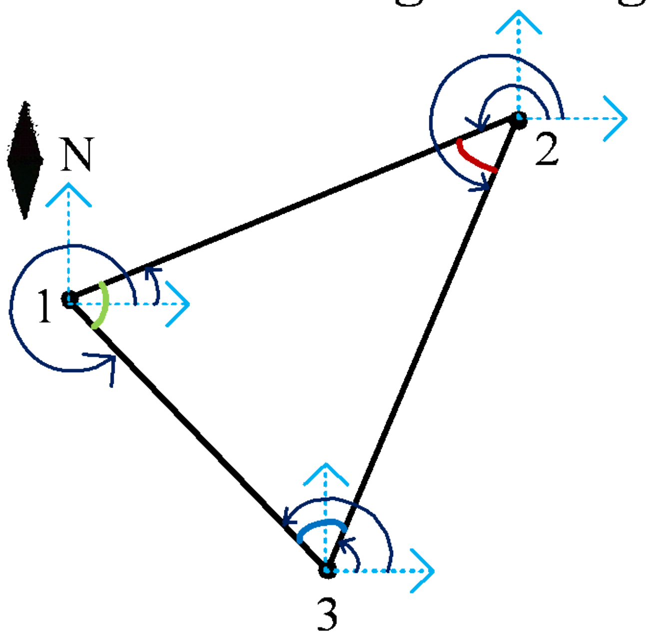
Bearing-based



Angle-based

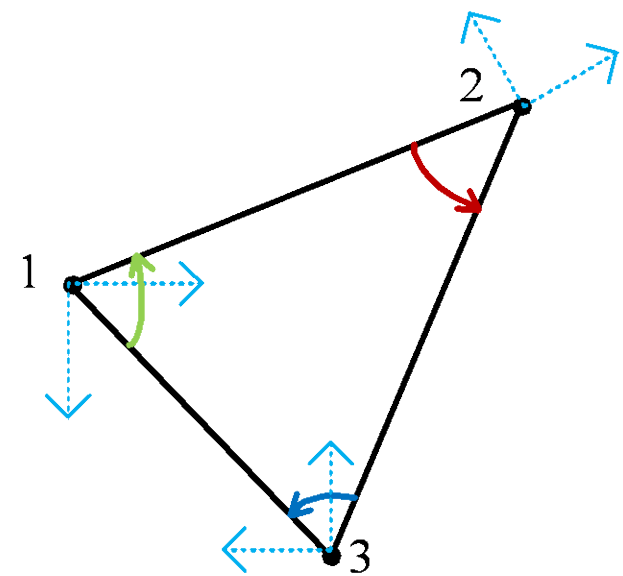
Formation maintenance under bearing/angle constraints

Formation using bearings



Aligned coordinate system

Formation using angles



Local coordinate system

Sensing in formation flying

F16 Thunderbirds

Roughly
- GPS like navigation;
- vision sensors;
- IMU-like sensors

Possibly
- vision;
- magnetite (like GPS);
- IMU-like sensors?

Sensing and measurements

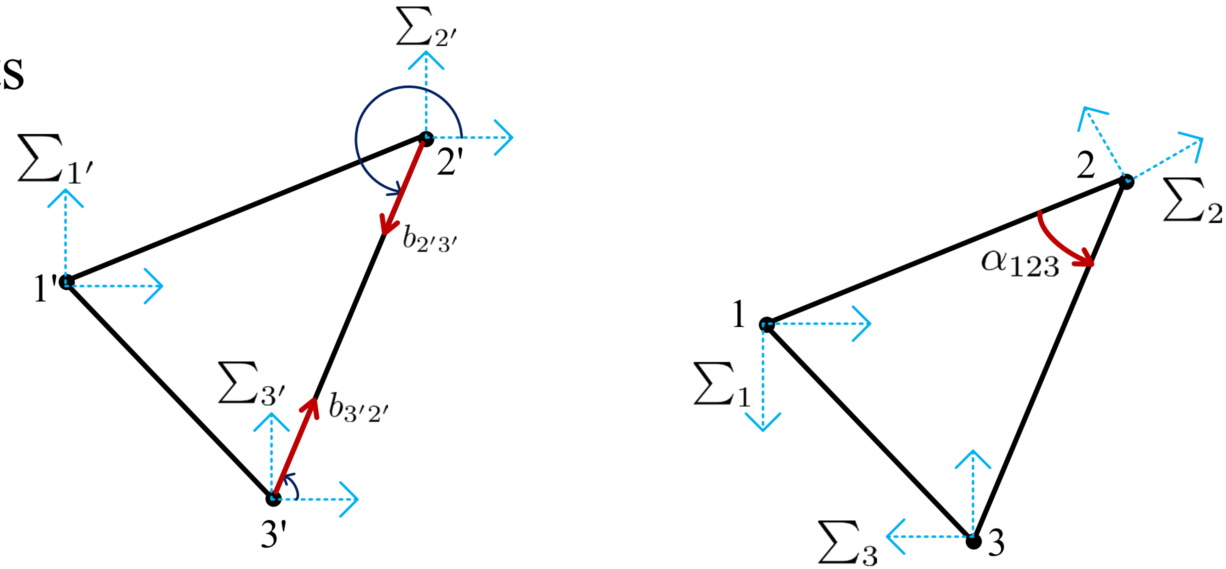
Approach / Property	Position	Displacement	Distance	Bearing
Shape description	Absolute Positions	Relative positions	Distances	Bearings
Example	$p_1^* = [0; 1]$	$(p_1 - p_2)^* = [0; 1]$	$\|p_1 - p_2\|^* = 1$	$\left(\frac{p_1 - p_2}{\|p_1 - p_2\|}\right)^* = [0; 1]$
Measurement	Absolute Position	Relative position	Local relative position	Bearing
Sensor (One case)	GPS receiver	IMU+compass +radar+camera	IMU+radar +camera	IMU+compass +camera



[1] Fax, J. A., & Murray, R. M. (2002). Graph laplacians and stabilization of vehicle formations. IFAC Proceedings Volumes.
[2] Anderson, B. D. O., Yu, C., Fidan, B., & Hendrickx, J. M. (2008). Rigid graph control architectures for autonomous formations. IEEE Control Systems Magazine
[3] Franchi, A., Masone, C., Grabe, V., Ryll, M., Bülthoff, H. H., & Giordano, P. R. (2012). Modeling and control of UAV bearing formations with bilateral high-level steering. The International Journal of Robotics Research.
[4] Zhao, S., & Zelazo, D. (2015). Bearing rigidity and almost global bearing-only formation stabilization. IEEE Transactions on Automatic Control.
[5] Michieletto, G., Cenedese, A., & Zelazo, D. (2021). A unified dissertation on bearing rigidity theory. IEEE Transactions on Control of Network Systems.

Angle meaurements

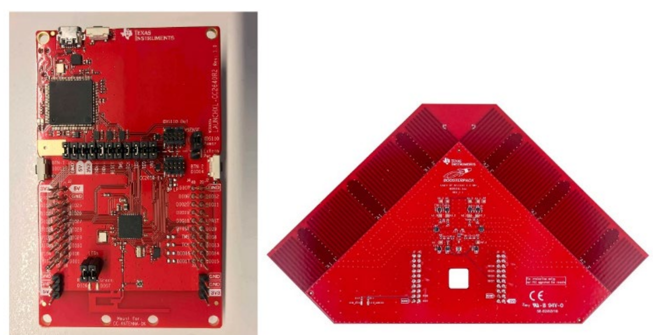
➢ Angle measurements



(a) Bearing measurements (b) Angle measurements

➢ sensors

Monocular camera with tag recognition [1]



Bluetooth 5.1-based AOA modules [2]

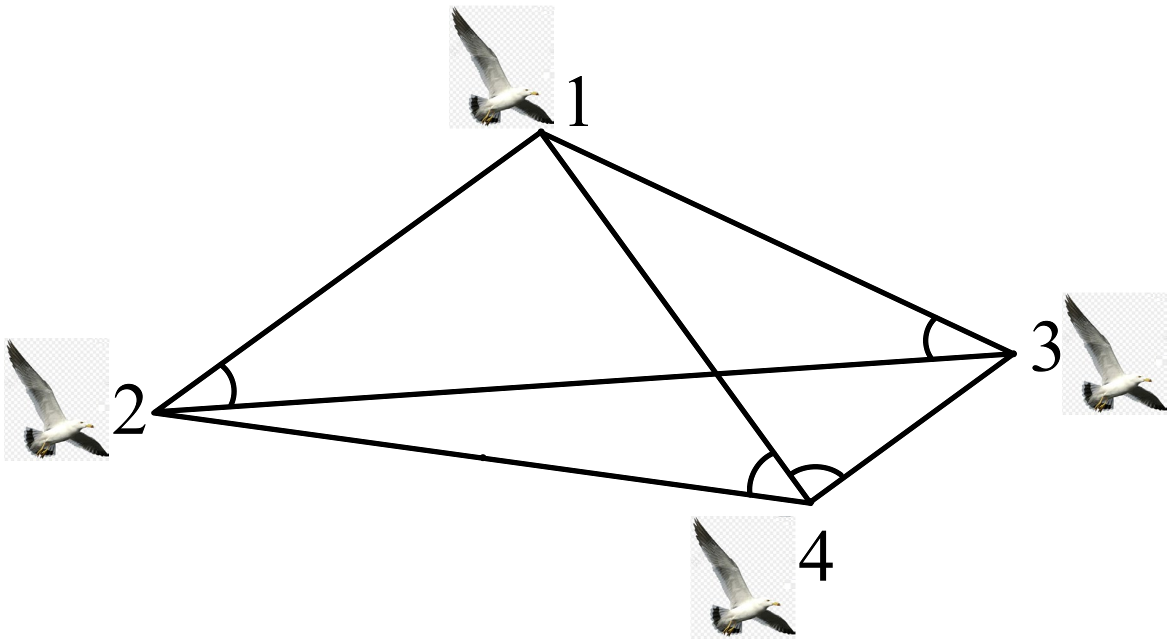
UWB-based AOA modules [3]

[1] Kamphuis M. (2020) Angle-based formation control applied on a team of Nexus robots, Master Thesis, University of Groningen
[2] Texas Instruments, AOA Receiver and AOA Transmitter
[3] UWB Shield and Antenna board

Outline

- Angle rigidity graph theory
  - Definitions
  - Construction methods
  - Checking conditions
- Rigidity of convex polyhedra
- Multi-agent formation control

Angle rigidity



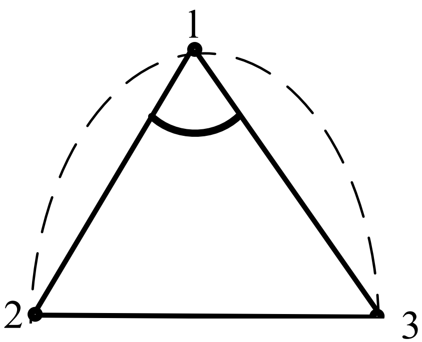
Research problems:

Under which set of angle constraints the shape of the formation is ***uniquely*** determined (rigid)?

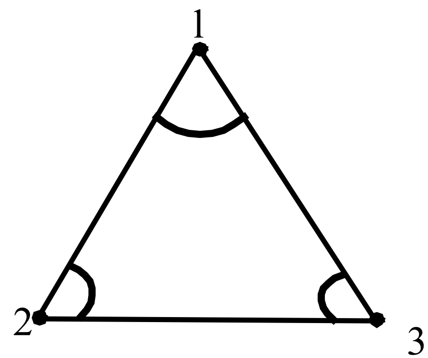
[1] Jing, G., Zhang, G., Lee, H. W. J., & Wang, L. (2019). Angle-based shape determination theory of planar graphs with application to formation stabilization. Automatica.
[2] Chen, L., Cao, M., & Li, C. (2021). Angle rigidity and its usage to stabilize multiagent formations in 2-D. IEEE Transactions on Automatic Control.
[3] Buckley, I., & Egerstedt, M. (2021). Infinitesimal shape-similarity for characterization and control of bearing-only multirobot formations. IEEE Transactions on Robotics.

Angle rigidity

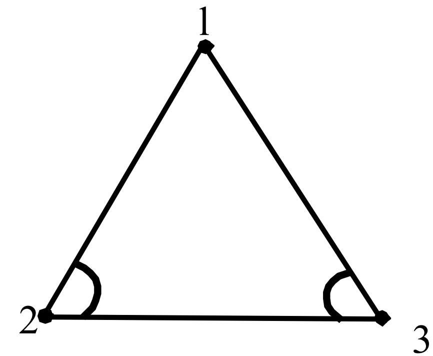
Starting from the 2D case:



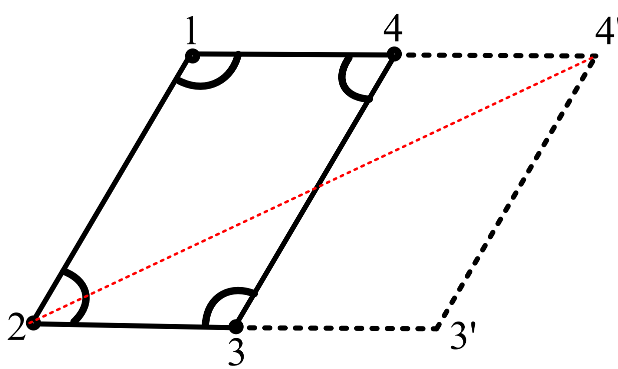
flexible



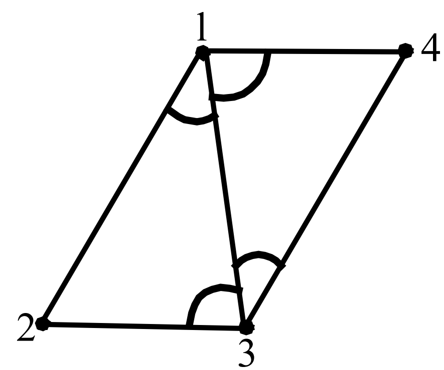
rigid



rigid



flexible

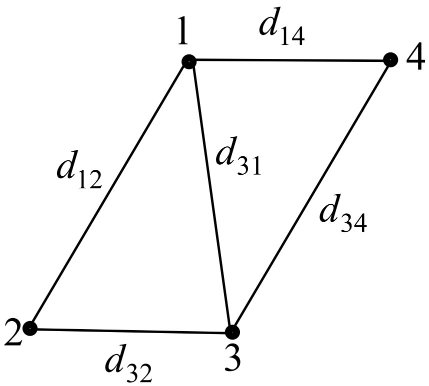


rigid

Definition

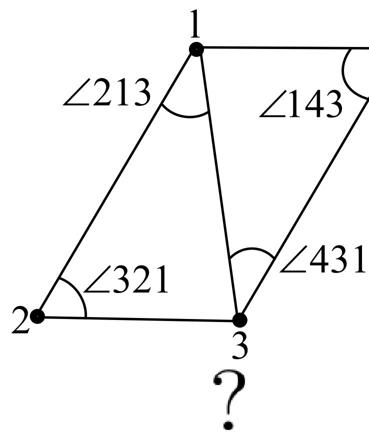
Angularity

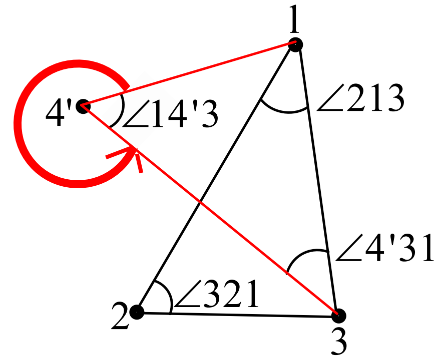
Distance rigidity



Graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$

Angle rigidity





Define ***angularity*** $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ = vertex set $\mathcal{V}$+ angle set $\mathcal{A}$+ position vector p

$$\mathcal{V} = \{1, 2, 3, 4\}, \;\; \mathcal{A} = \{(2, 1, 3), (3, 1, 4), (1, 3, 2), (1, 3, 4)\}, \;\; p = [p_1^T, p_2^T, p_3^T, p_4^T]^T$$

$\measuredangle jik \in [0, 2\pi)$ *counterclockwise* $\qquad \measuredangle 143 = 60^\circ \qquad \measuredangle 14'3 = 360^\circ - 60^\circ$

[1] S. Franco, & W. Whiteley, Constraining plane configurations in CAD: circles, lines, and angles in the plane. SIAM Journal on Discrete Mathematics, 2004.
[2] G. Jing, G. Zhang, H. W. J. Lee, & L. Wang, Angle-based shape determination theory of planar graphs with application to formation stabilization, Automatica, 2019.
[3] L. Chen, M. Cao, & C. Li, Angle rigidity and its usage to stabilize multi-agent formations in 2D, IEEE Trans. Automat. Contr., 2021.

Definition

Angle rigidity

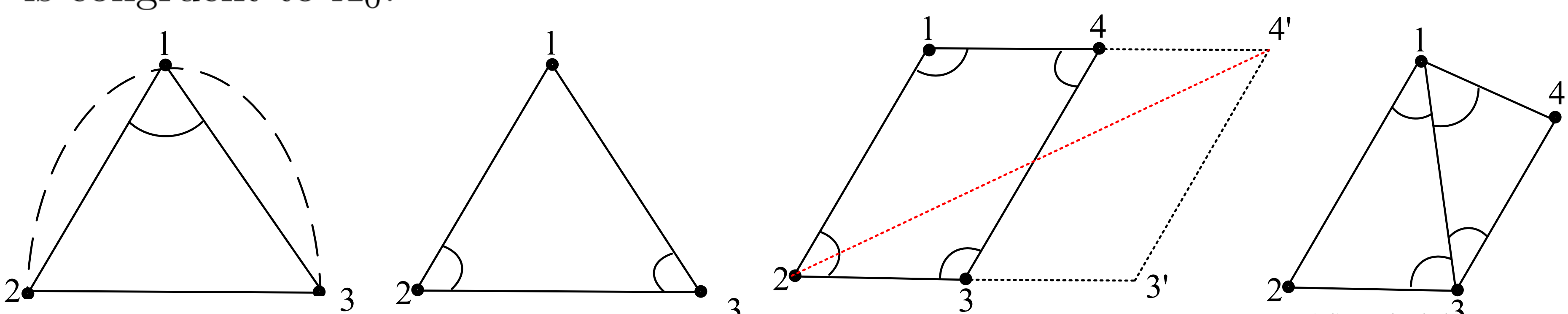
We say two angularities $\mathbb{A}_0(\mathcal{V}, \mathcal{A}, p)$ and $\mathbb{A}_1(\mathcal{V}, \mathcal{A}, p')$ with the same $\mathcal{V}$ and $\mathcal{A} \subset \mathcal{V} \times \mathcal{V} \times \mathcal{V} = \{(i,j,k), i,j,k \in \mathcal{V}, i \neq j \neq k\}$ are ***equivalent*** if

$$\measuredangle ijk(p_i, p_j, p_k) = \measuredangle ijk(p_i', p_j', p_k') \text{ for } (i,j,k) \in \mathcal{A}.$$

We say they are ***congruent*** if

$$\measuredangle ijk(p_i, p_j, p_k) = \measuredangle ijk(p_i', p_j', p_k') \text{ for all } i,j,k \in \mathcal{V}, or\ (i,j,k) \in \mathcal{A}^*.$$

An angularity $\mathbb{A}_0(\mathcal{V}, \mathcal{A}, p)$ is ***angle rigid*** if there exists an $\epsilon > 0$ such that every angularity $\mathbb{A}_1(\mathcal{V}, \mathcal{A}, p')$ that is equivalent to $\mathbb{A}_0$ and satisfies $\|p' - p\| < \epsilon$, is congruent to $\mathbb{A}_0$.



(a) Non-rigid (b) Rigid (c) Non-rigid (d) Rigid

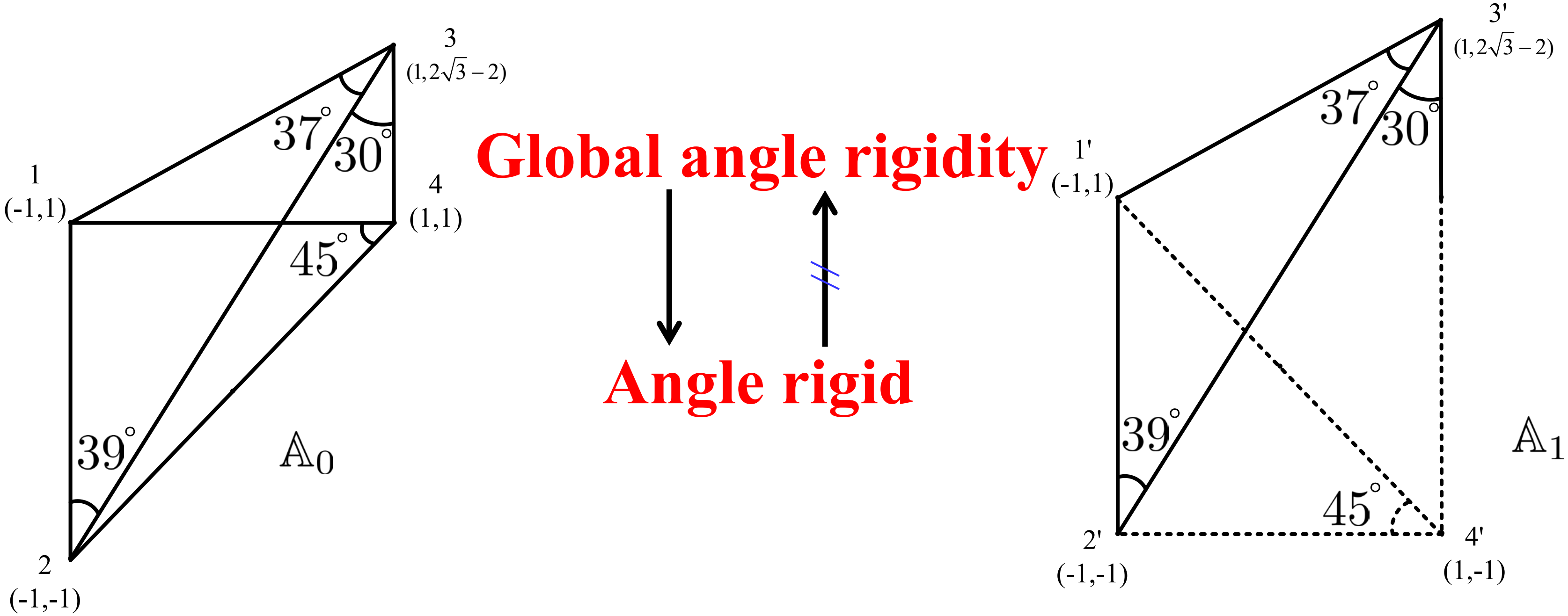
[1] L. Chen, M. Cao, & C. Li, Angle rigidity and its usage to stabilize multi-agent formations in 2D, IEEE Trans. Automat. Contr., 2021.

Definition

Global angle rigidity

An angularity $\mathbb{A}_0(\mathcal{V}, \mathcal{A}, p)$ is *angle rigid* if there exists an $\epsilon > 0$ such that every angularity $\mathbb{A}_1(\mathcal{V}, \mathcal{A}, p')$ that is equivalent to $\mathbb{A}_0$ and satisfies $\|p' - p\| < \epsilon$, is congruent to $\mathbb{A}_0$.

If this satisfies for all $\epsilon \in \mathbb{R}$, $\mathbb{A}_0(\mathcal{V}, \mathcal{A}, p)$ is ***globally angle rigid***.



$$\mathbb{A}_0\left(\{1,2,3,4\},\quad \{(2,1,3),(3,1,4),(1,3,2),(1,3,4)\},\quad [p_1^T, p_2^T, p_3^T, p_4^T]^T\right)$$

Small perturbation — Large perturbation

Angle rigid — NOT globally angle rigid

[1] L. Chen, M. Cao, &C. Li, Angle rigidity and its usage to stabilize multi-agent formations in 2D, IEEE Trans. Automat. Contr., 2021.

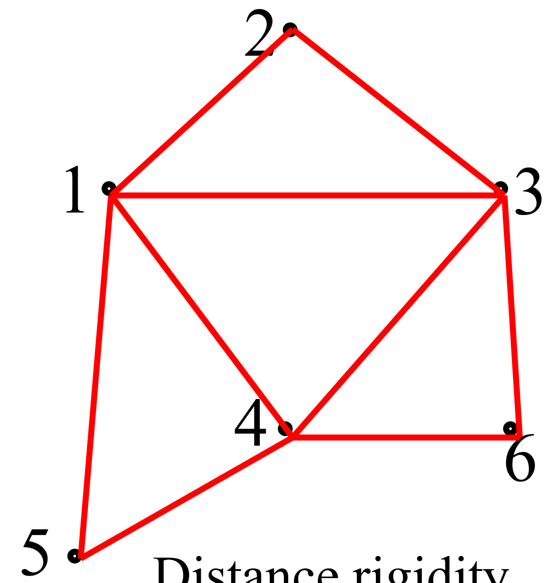
Construction method

Angle rigidity:

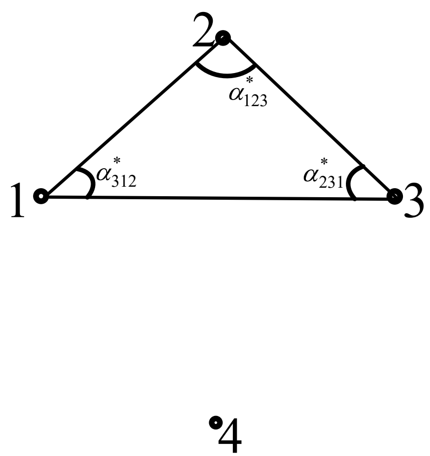
- Step 1: Start from a triangular shape
- Step 2: Add vertex 4 by two angles

Case 1: $\alpha_{214}, \alpha_{234}$ (Globally angle rigid)

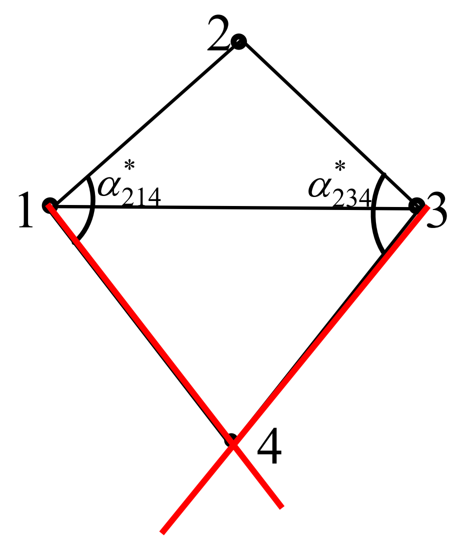
Case 2: $\alpha_{214}, \alpha_{243}$?



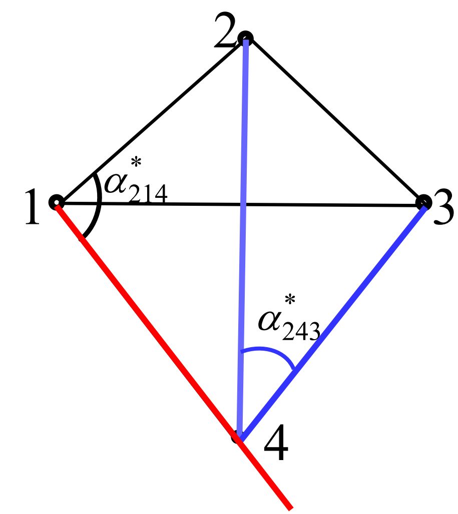
Distance rigidity



Step 1



Step 2: Case 1



Step 2: Case 2

[1] L. Henneberg, *Die Graphische Statik der starren Systeme.* Leipzig: B.G. Teubner, 1911.

[2] L. Chen, M. Cao, &C. Li, Angle rigidity and its usage to stabilize multi-agent formations in 2D, IEEE Trans. Automat. Contr., 2021.

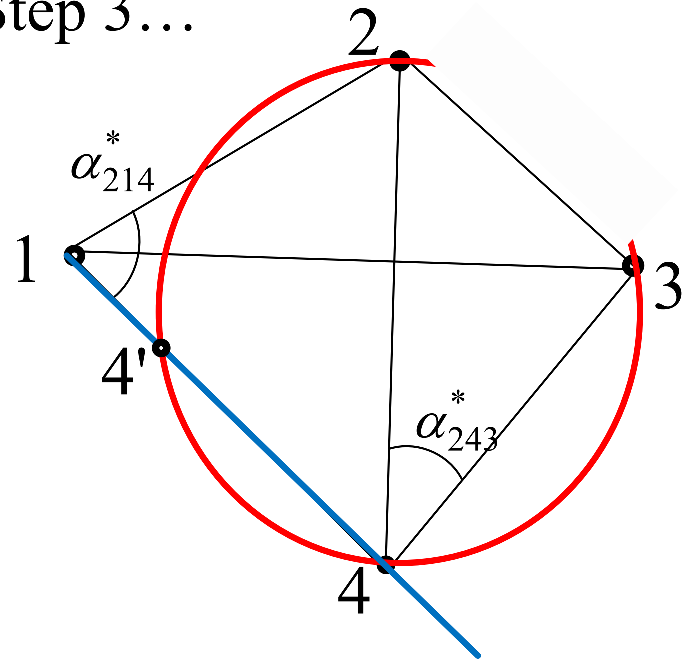
Construction method

- Step 1: Start from a triangular shape
- Step 2: Add vertex 4 by two angles

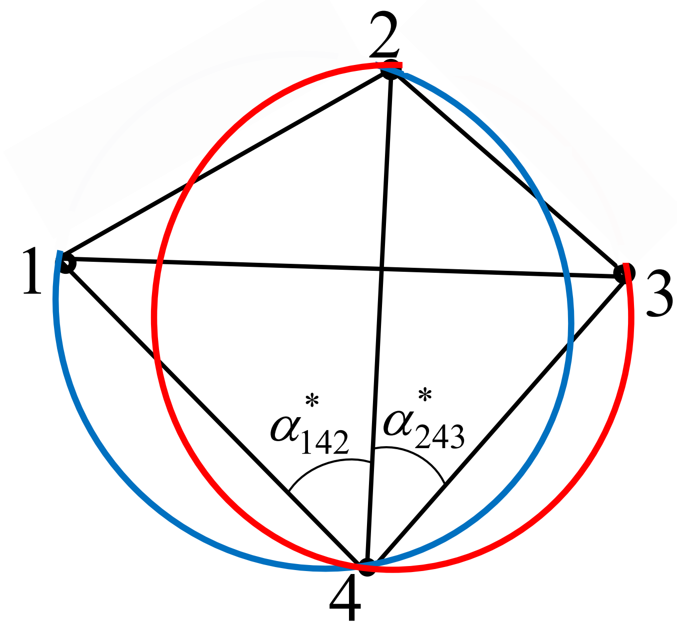
  Case 1: $\alpha_{214}, \alpha_{234}$ (Globally angle rigid): Type I

  Case 2: $\alpha_{214}, \alpha_{243}$ (Angle rigid): Type II

  Case 3: $\alpha_{142}, \alpha_{243}$ (Globally angle rigid): Type I
- Step 3…



Step 2: Case 2



Step 2: Case 3

Checking condition

$\mathbb{A}(\mathcal{V}, \mathcal{A}, p) \longrightarrow$

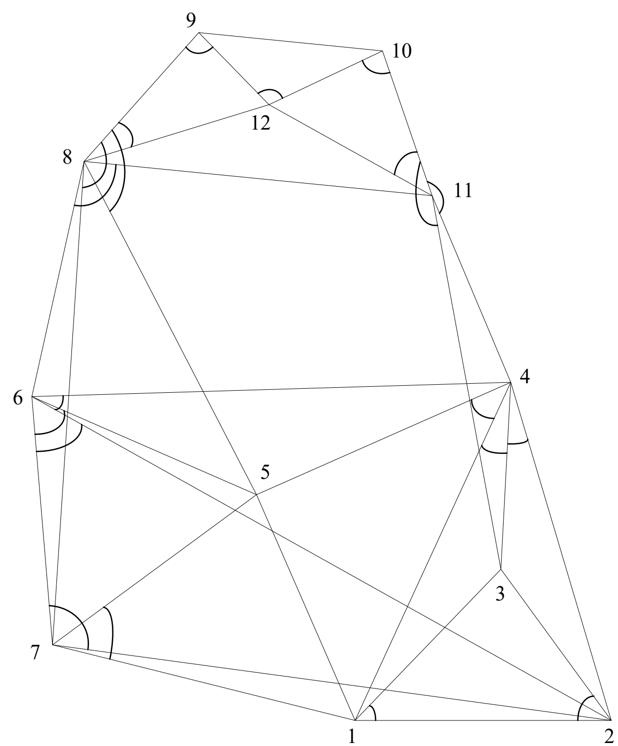
Angle rigid?

① Algebraic: $\text{Rank}(R_a(p)) = 2|\mathcal{V}| - 4$

② Topological: $\mathcal{A}$ contains a Type-II construction.

Globally angle rigid?

① Topological: $\mathcal{A}$ contains a Type-I construction.



Main challenge: For a minimally angle rigid angularity, each vertex can be associated with 5 angle constraints

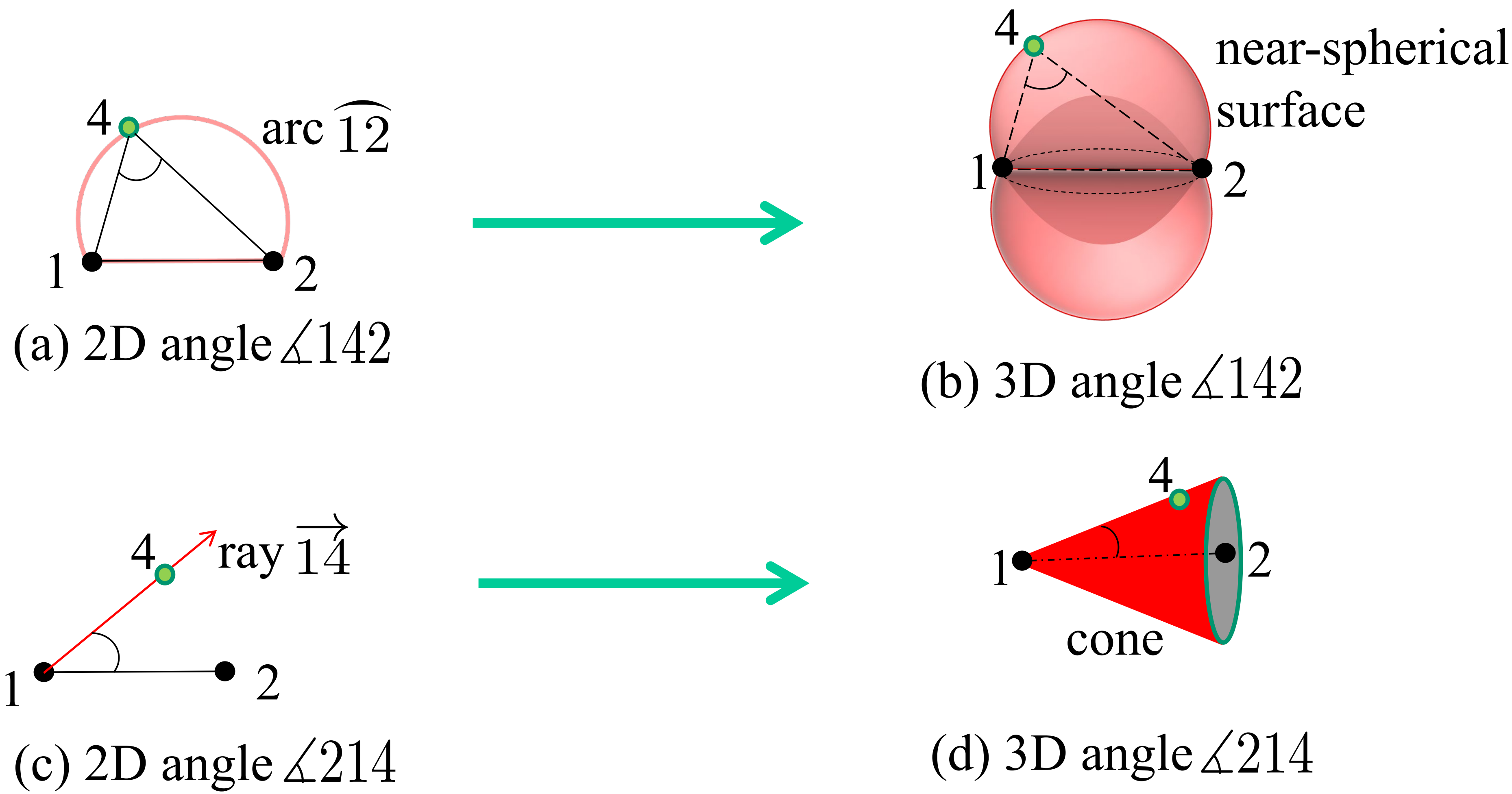
More complicated than distance rigidity case

Angle rigidity's topological, necessary and sufficient conditions are still unknown

[1] G. Laman, "On graphs and rigidity of plane skeletal structures," Journal of Engineering mathematics, pp. 331–340, 1970.

[2] L. Chen, M. Cao, &C. Li, Angle rigidity and its usage to stabilize multi-agent formations in 2D, IEEE Trans. Automat. Contr., 2021.

Extension to 3D



- 3D Angle rigidity's construction methods and checking conditions can be developed.

Rigidity of convex polyhedra?

Outline

- Angle rigidity graph theory
  - Definitions
  - Construction methods
  - Checking conditions

- Rigidity of convex polyhedra

- Multi-agent formation control

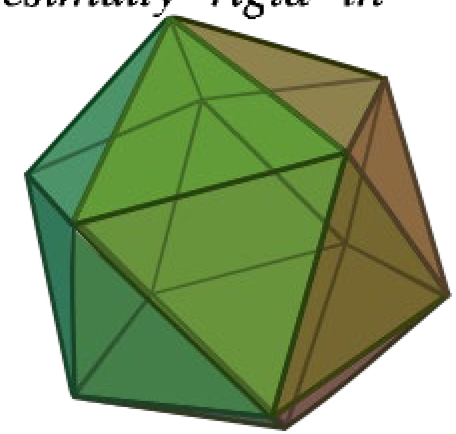
Background

- Cauchy's rigidity theorem for 3-dimensional polyhedral[1]

Theorem. *If two 3-dimensional convex polyhedra P and P' are combinatorially equivalent with corresponding facets being congruent, then also the angles between corresponding pairs of adjacent facets are equal (and thus P is congruent to P').*

- Rigidity theorem for distance-constrained convex polyhedra by Dehn, Aleksandrov, Gluck, etc[2]

Theorem 4.4. *Let P be a compact convex polytope in three-space with all faces triangles. Then the associated bar framework $G(\mathbf{p})$ is infinitesimally rigid in three-space.*



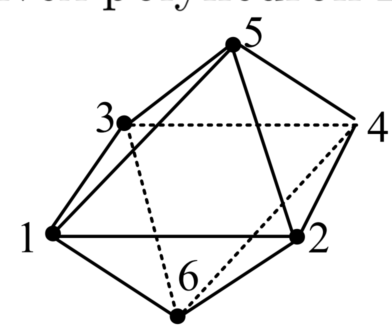
How about angle constraints ?

[1] Aigner, Martin; Ziegler, Günter M. (2014). Proofs from THE BOOK. Springer. pp. 71–74. ISBN 9783540404606.
[2] Connelly, R. (1993). Rigidity. In Handbook of convex geometry (pp. 223-271). North-Holland.

Polyhedra with triangular faces

- Rigidity theorem for angle-constrained convex polyhedra

Theorem The angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ obtained from a convex polyhedron $\mathbb{P}$ with all faces being triangles is angle rigid.



(a) Convex polyhedron with triangular surfaces

Lemmas for the proof of the theorem:

Lemma 1[1] If all angles on the faces of a convex polyhedron $\mathbb{P}$ remain constant when $\mathbb{A}$ is perturbed, then all the dihedral angles of $\mathbb{P}$ remain constant.

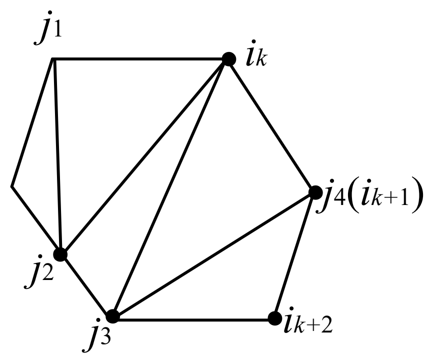
Lemma 2[1] If all edge lengths, angles in faces and dihedral angles of a convex polyhedron $\mathbb{P}$ remain constant under a perturbation of $\mathbb{A}$, then the perturbation must be a translation or rotation of $\mathbb{A}$.

[1] Alexandrov, A. D. (2005). Convex polyhedra (Vol. 109). Berlin: Springer.

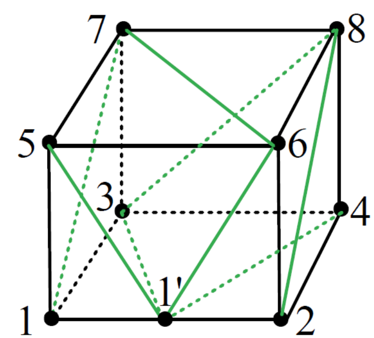
Polyhedra with polygonal faces

Definition 1 (Polygonal triangulation[1]) Polygonal triangulation is the decomposition of a polygon into a set of triangles where any two of these triangles either do not intersect at all or intersect at a common vertex or edge.

Definition 2 (Surface triangulation) Surface triangulation for a polyhedron $\mathbb{P}$ is the decomposition of the surface of $\mathbb{P}$ using polygonal triangulation for each face of $\mathbb{P}$ and at the same time any two decomposed triangles from two faces of $\mathbb{P}$ either do not intersect at all or intersect at a common vertex or edge.



(a) Polygonal triangulation



(b) Surface triangulation

Theorem A convex triangulated polyhedral angularity $\mathbb{A}(\mathcal{V} \cup \mathcal{V}', \mathcal{A}, [p^\top, p'^\top]^\top)$ without any vertex of $\mathcal{V}'$ lying in the interior of a face of $\mathbb{P}$ is angle rigid.

[1] Connelly, R. (1993). Rigidity. In Handbook of convex geometry (pp. 223-271). North-Holland.

Outline

- Angle rigidity graph theory
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- Rigidity of convex polyhedra

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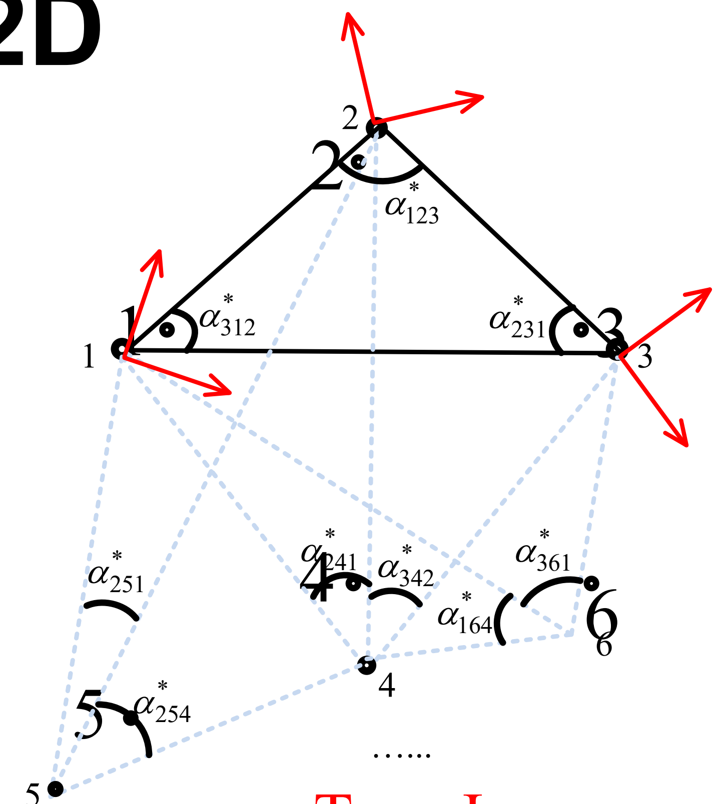
Formation control in 2D



Type-I

$$\dot{p}_i = u_i, i = 1, \cdots, N,$$

$$\alpha_{jik} = \arccos(b_{ij}^T b_{ik})$$

$$b_{ij} = \frac{p_j - p_i}{\|p_j - p_i\|}, j \in \mathcal{N}_i$$

Problem 1 *Given feasible desired angles*

$$f_{\mathcal{A}} = \{\alpha^*_{312}, \alpha^*_{123}, \alpha^*_{231}, \alpha^*_{241}, \alpha^*_{342}, \cdots, \alpha^*_{i_1 N i_2}, \alpha^*_{i_2 N i_3}\} \tag{1}$$

design control law u_i by using local direction measurements $b_{ij}, j \in \mathcal{N}_i$ to achieve

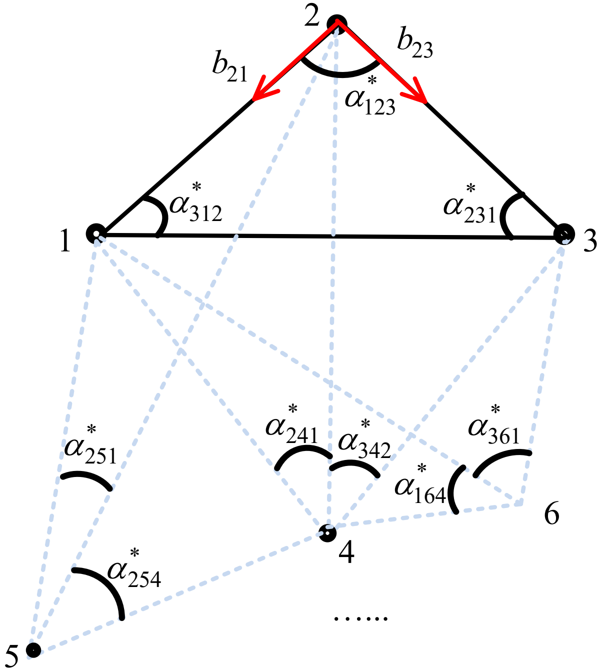
$$\lim_{t\to\infty}(\alpha_{jik}(t) - \alpha^*_{jik}) = 0, \ (j, i, k) \in \mathcal{A} \tag{2}$$

[1] L. Chen, M. Cao, &C. Li, Angle rigidity and its usage to stabilize multi-agent formations in 2D, IEEE Trans. Automat. Contr., 2021.

Formation control in 2D

$$V = \sum_{(j,i,k)\in\mathcal{A}} (\alpha_{jik} - \alpha_{jik}^*)^2$$

$$u_i = -(\frac{\partial V}{\partial p_i})^T = f(l_{ij}, b_{ij}, l_{jk}, b_{jk})$$



$$\dot{p}_i = u_i = -\sum_{(j,i,k)\in\mathcal{A}} (\alpha_{jik} - \alpha_{jik}^*)\underline{(b_{ij} + b_{ik})}.$$

Bisector moving rule

$$b_{ij} = \frac{p_j - p_i}{\|p_j - p_i\|}, \alpha_{jik} = \arccos(b_{ij}^T b_{ik})$$

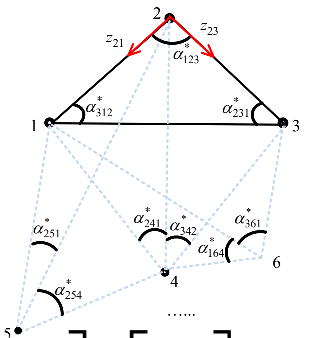
$$u_1 = -(\alpha_1 - \alpha_1^*)(b_{12} + b_{13})$$
$$u_2 = -(\alpha_2 - \alpha_2^*)(b_{21} + b_{23})$$
$$u_3 = -(\alpha_3 - \alpha_3^*)(b_{31} + b_{32})$$
$$u_4 = -(\alpha_{241} - \alpha_{241}^*)(b_{41} + b_{42}) - (\alpha_{342} - \alpha_{342}^*)(b_{42} + b_{43})$$
$$u_5, u_6, \cdots\cdots$$

$$e_2 = \alpha_2 - \alpha_2^*$$

$\alpha_2(0) < \alpha_2^*$	$\alpha_2(0) > \alpha_2^*$
Move inwards	Move outwards
$\alpha_2 \uparrow$	$\alpha_2 \downarrow$

$$|e_2| \downarrow$$

Formation control in 2D



Stability analysis

$$\dot{e}_a = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_{41} \\ \dot{e}_{42} \\ \cdots \\ \dot{e}_{N2} \end{bmatrix} = A(e_a)e_a = \begin{bmatrix} F_1(e_s) & 0 & 0 & \cdots & 0 \\ H_4(e_4) & F_4(e_4) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_i(e_i) & \cdots & F_i(e_i) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_N(e_N) & \cdots & \cdots & \cdots & F_N(e_N) \end{bmatrix} \begin{bmatrix} e_s \\ e_4 \\ \cdots \\ e_i \\ \cdots \\ e_N \end{bmatrix}$$

Theorem 1 The nonlinear angle error dynamics $\dot{e}_a = A(e_a)e_a$ is locally exponentially stable around the desired equilibrium $e_a = 0$.

Proof Linearization $\rightarrow \frac{\partial[A(e_a)e_a]}{\partial e_a}|_{e_a=0}$ is Hurwitz.

Theorem 2 The first three agents' angle error dynamics $\dot{e}_s = F_1(e_s)e_s$ is almost globally stable.

Proof Poincare-Bendixson Theorem.

[1] L. Chen, M. Cao, &C. Li, Angle rigidity and its usage to stabilize multi-agent formations in 2D, IEEE Trans. Automat. Contr., 2021.

Formation control in 2D

Problem 2

$$\dot{p}_i = u_i, \quad \lim_{t\to\infty}(\alpha_{jik}(t) - \alpha^*_{jik}) = 0, \quad \lim_{t\to\infty}(\dot{p}_i(t) - \underline{v^*_t(t) - v^*_r(t) - v^*_s(t)}) = 0$$

Desired translational, rotational, and scaling velocity

$$\begin{aligned} u_i =& -k_i(\alpha_i - \alpha_i^* - \boxed{\frac{\mu_i}{k_i}})b_{i(i+1)} - k_i(\alpha_i - \alpha_i^* - \boxed{\frac{\tilde{\mu}_i}{k_i}})b_{i(i-1)} \\ =& -k_i(\alpha_i - \alpha_i^*)[b_{i(i+1)} + b_{i(i-1)}] + [\mu_i b_{i(i+1)} + \tilde{\mu}_i b_{i(i-1)}] \\ =& u_{fi} + u_{mi} \end{aligned} \quad (1)$$

Problem 3

$$\ddot{p}_i = u_i, \quad \lim_{t\to\infty}(\alpha_{jik}(t) - \alpha^*_{jik}) = 0, \quad \lim_{t\to\infty}\dot{p}_i(t) = 0$$

$$u_i = -k_s\dot{p}_i - \sum_{(j,i,k)\in\mathcal{A}}(\alpha_{jik} - \alpha^*_{jik})(b_{ij} + b_{ik}) \quad (2)$$

[1] L. Chen, H. Garcia de Marina, M. Cao, Maneuvering formations of mobile agents using designed mismatched angles. IEEE Trans. Automat Contr., 2021.

[2] L. Chen, M. Shi, H. Garcia de Marina, M. Cao, Stabilizing and maneuvering angle rigid multi-agent formations with double-integrator agent dynamics, IEEE Trans. Control of Network Systems, 2022.

Formation control in 3D

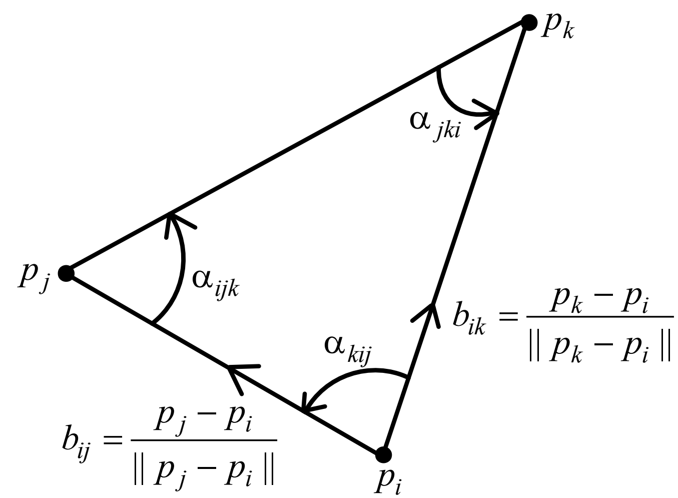
First three agents

Locally stable control law

A convex combination

$$u_i(t) = -(\alpha_{[i-1]i[i+1]}(t) - \alpha^*_{[i-1]i[i+1]})(\gamma_1 b_{i[i-1]}(t) + \gamma_2 b_{i[i+1]}(t)), i = 1, 2, 3$$

where $\gamma_1 \geq 0, \gamma_2 \geq 0$ and $\gamma_1 + \gamma_2 = 1.$



Globally stable control law

$$u_1 = 0,$$

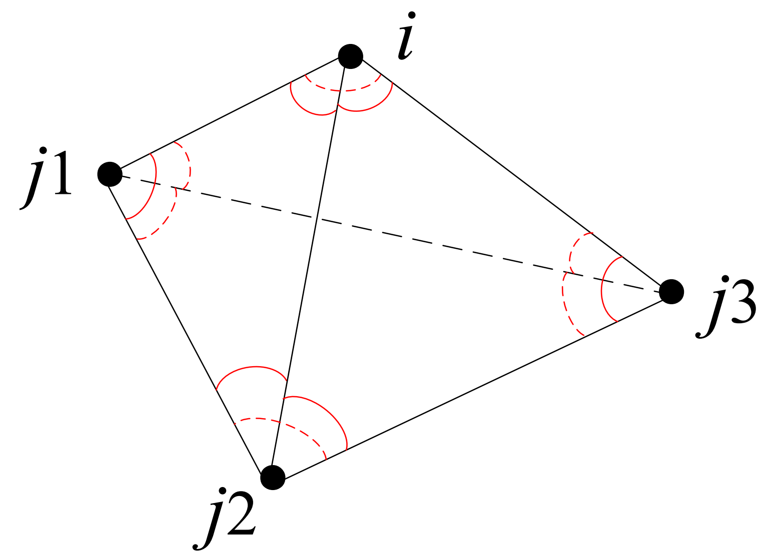
$$u_2 = -(\alpha_{123} - \alpha^*_{123})b_{23}, \quad u_3 = -(\alpha_{231} - \alpha^*_{231})b_{32}$$

Formation control in 3D

The remaining agents

Type I Pursuing rule → More efficient

$$u_i = (\alpha_{ij_1j_2} - \alpha^*_{ij_1j_2})\underline{b_{ij_2}} + (\alpha_{ij_2j_1} - \alpha^*_{ij_2j_1})b_{ij_1} + (\alpha_{ij_2j_3} - \alpha^*_{ij_2j_3})b_{ij_3}, \quad 4 \leq i \leq N$$



Type-II Bisector moving rule

$$u_i = -(\alpha_{j_1ij_2} - \alpha^*_{j_1ij_2})\underline{(b_{ij_1} + b_{ij_2})} - (\alpha_{j_2ij_3} - \alpha^*_{j_2ij_3})(b_{ij_2} + b_{ij_3}) - (\alpha_{j_3ij_1} - \alpha^*_{j_3ij_1})(b_{ij_3} + b_{ij_1}), \quad 4 \leq i \leq N$$

Outline

- Angle rigidity graph theory
  - Definitions
  - Construction methods
  - Checking conditions
- Rigidity of convex polyhedra
- Multi-agent formation control

Concluding remarks

- Formation flying for robotic teams relies on the enabling ***sensing*** technology.
- Different rigidity properties of formations arise when the constraints are in terms of positions, angles and bearings.
- Sufficient conditions can be established for angle and global angle rigidity.
- Formation control laws can be further developed with the help of angle rigidity graph theory.

Some selected recent publications from my group on related topics

Angle rigidity and its usage for formation maneuvering:

"Angle rigidity and its usage to stabilize multi-agent formations in 2D," L. Chen, M. Cao and C. Li*. IEEE Trans. on Automatic Control,* V66, Issue 8, 3667-3681, 2020

"Maneuvering formations of mobile agents using designed mismatched angles," L. Chen, H. Garcia de Marina, and M. Cao*. IEEE Trans. on Automatic Control,* V67, Issue 4, 1655-1668, 2021

"Stabilizing and maneuvering angle rigid multi-agent formations with double-integrator agent dynamics," L. Chen, M. Shi, H. Garcia de Marina and M. Cao*.* *To appear, IEEE Trans. on Control of Network Systems*, 2022

"Angle rigidity for multi-agent formations in 3D," L. Chen and M. Cao*. IEEE Trans. on Automatic Control*, conditionally accepted, 2022

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